A Nonlinear Stability Analysis of a Double-Diffusive Magnetized Ferrofluid

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A nonlinear (energy) stability analysis is performed for a magnetized ferrofluid layer, heated and soluted from below, with stress-free boundaries. A rigorous nonlinear stability result is derived by introducing a suitable generalized energy functional. The mathematical emphasis is on how to control the nonlinear terms caused by magnetic body and inertia forces. For ferrofluids we find that the existence of subcritical instabilities is possible, however, it is noted that, in case of a nonferrofluid, the global nonlinear stability Rayleigh number is exactly the same as that for linear instability. For lower values of the magnetic parameters, this coincidence is immediately lost. The effects of the magnetic parameter, M_3, and the solute gradient, S_1, on the subcritical instability region have also been analyzed. It is shown that with the increase of the magnetic parameter the subcritical instability region between the two theories decreases quickly, while with the increase of the solute gradient the subcritical region expands. We also demonstrate the coupling between the buoyancy and magnetic forces in the nonlinear energy stability analysis.

Key words: Magnetized Ferrofluid; Nonlinear Stability; Double-Diffusive Convection; Magnetization.

1. Introduction

The energy method and the linear stability theory complement each other in demarking the range of parameter space in which subcritical instabilities might arise. The energy method is essentially due to Orr [1], and its recent revival has been inspired by the work of Serrin [2] and Joseph [3 – 5]. Rapid improvements of the classical energy theory have been made in recent years [6]. By introducing the coupling parameters into the energy method and by selecting them optimally, it has been possible to sharpen the stability bound in many physical problems [7]. A nonlinear stability analysis of non-magnetic fluids by using the generalized energy stability theory has been considered by many authors [8 – 14]. Ferrofluids are non-naturally occurring magnetic suspensions obtained by seeding surfactant-wrapped single-domain superparamagnetic nanoparticles in an appropriate organic solvent [15, 16]. The theory of convective instability of ferrofluids began with Finlayson [17] and interestingly continued by Lalas and Carmi [18], Shliomis [19], Schwab et al. [20], Stiles and Kagan [21], Blennerhassett et al. [22], Venkatasubramanian and Kaloni [23] and Sunil and co-workers [24 – 26]. Recently, Sunil and Mahajan [27] studied the nonlinear stability analysis of magnetized ferrofluids heated from below.

A really interesting situation, from a mathematical viewpoint, arises when the layer is simultaneously heated and salted from below. In the standard Bénard problem, the instability is driven by a density difference caused by a temperature difference between the upper and lower planes bounding the fluid. If in the fluid layer additionally salt is dissolved then there are potentially two destabilizing sources for the density difference, the temperature field and the salt field. When there are two effects such as these, the phenomenon of convection which arises is called double-diffusive convection. The driving force for many studies on double-diffusive or multi-component convection are largely physical applications. Double-diffusive convection problems have been studied by many authors [28 – 36]. Such a fluid is applicable as a working fluid in absorption refrigeration, as a solution in electro- or electroless plating, and as a transfer medium in medical treatment.

In the present paper, we study the nonlinear stability analysis of a magnetized ferrofluid, heated and soluted from below, via a generalized energy method. When buoyancy magnetization is absent (non-ferrofluid), then there is coincidence between the nonlinear and linear stability results. This in turn implies exclusion of the occurrence of subcritical instability. For convection problems in magnetized ferrofluids, the linear critical magnetic thermal Rayleigh number is found to be higher than the nonlinear (energy) critical magnetic thermal Rayleigh number, which shows the possibility of the existence of subcritical instability. Comparison of the results obtained, respectively, by the energy method and the linear stability analysis are discussed finally. We also examine the coupling between the buoyancy and magnetic forces in the nonlinear stability analysis. This problem, to the best of our knowledge, has not been investigated yet. Borglin et al. [37] has discussed the experimental studies of flow of ferrofluids in porous medium. They have presented laboratory-scale experimental results of the behaviour of ferrofluids in a porous medium consisting of sands and sediments. We believe that the present study can serve as a theoretical support for experimental investigations in double-diffusive ferroconvection. To date, however, no experimental study on double-diffusive magnetized ferrofluids has been performed.

2. Mathematical Formulation of the Problem

We consider an electrically non-conducting incompressible thin magnetized ferrofluid containing two diffusive components, which is occupying an infinite horizontal layer between $z=-\frac{d}{2}$ and $z=+\frac{d}{2}$, with the temperature of the lower plane maintained at T_L and the temperature of the upper plane maintained at T_U, such that $T_L > T_U$. The gravity field is acting in the negative z-direction and the magnetic field, $\boldsymbol{H}=H_0^{\text{ext}}\hat{\boldsymbol{k}}$, acts outside the layer.

To describe nonlinear energy stability results in double-diffusive convection, we begin by introducing the relevant equations (utilizing the Boussinesq approximation), as given by the following [17, 27, 32]):

$$\nabla\cdot\boldsymbol{q}=0, \tag{1}$$

$$\begin{aligned}\rho_0\left(\frac{\partial}{\partial t}+\boldsymbol{q}\cdot\nabla\right)\boldsymbol{q}&=-\nabla p+\mu\nabla^2\boldsymbol{q}\\&+\rho_0\left[1-\alpha(T-T_a)+\alpha'(C-C_a)\right]\boldsymbol{g}\\&+\mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H},\end{aligned} \tag{2}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{q}\cdot\nabla\right)T=\kappa\nabla^2 T, \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{q}\cdot\nabla\right)C=\kappa'\nabla^2 C, \tag{4}$$

$$\nabla\cdot\boldsymbol{B}=0,\quad \nabla\times\boldsymbol{H}=\boldsymbol{0},\quad \boldsymbol{B}=\mu_0(\boldsymbol{H}+\boldsymbol{M}), \tag{5}$$

$$\begin{aligned}\boldsymbol{M}=\frac{\boldsymbol{H}}{H}\big[&M_0+\chi(H-H_0)-K_1(T-T_a)\\&+K_2(C-C_a)\big].\end{aligned} \tag{6}$$

Here, ρ, ρ_0, $\boldsymbol{q}$, t, p, μ, μ_0, $\boldsymbol{M}$, $\boldsymbol{B}$, κ, κ', α, and α' are the fluid density, reference density, velocity, time, pressure, viscosity, magnetic permeability of vacuum, magnetization, magnetic induction, thermal diffusivity, solute diffusivity, thermal expansion coefficient, and concentration expansion coefficient analogous to the thermal expansion coefficient, respectively. T_a and C_a are the average temperature and solute concentration given by $T_a=\frac{(T_L+T_U)}{2}$, $C_a=\frac{(C_L+C_U)}{2}$, respectively, where C_L and C_U are the solute concentrations of the lower and upper surfaces of the layer, $\beta(=|\mathrm{d}T/\mathrm{d}z|)$ and $\beta'(=|\mathrm{d}C/\mathrm{d}z|)$ are uniform temperature and solute gradients, respectively, $H=|\boldsymbol{H}|$, $M=|\boldsymbol{M}|$, and $M_0=M(H_0,T_a,C_a)$. The magnetic susceptibility, pyromagnetic coefficient, and salinity magnetic coefficient are defined by

$$\chi\equiv\left(\frac{\partial M}{\partial H}\right)_{H_0,T_a,C_a},$$
$$K_1\equiv-\left(\frac{\partial M}{\partial T}\right)_{H_0,T_a,C_a},\ \text{and}$$
$$K_2\equiv\left(\frac{\partial M}{\partial C}\right)_{H_0,T_a,C_a},\ \text{respectively.}$$

The basic state is assumed to be the quiescent state given by

$$\begin{aligned}&\boldsymbol{q}=\boldsymbol{q}_b=(0,\ 0,\ 0),\\&\rho=\rho_b(z)=\rho_0(1+\alpha\beta z-\alpha'\beta' z),\quad p=p_b(z),\\&T=T_b(z)=-\beta z+T_a,\quad C=C_b(z)=-\beta' z+C_a,\\&\beta=\frac{T_L-T_U}{d},\quad \beta'=\frac{C_L-C_U}{d},\\&\boldsymbol{H}_b=\left[H_0-\frac{K_1\beta z}{1+\chi}+\frac{K_2\beta' z}{1+\chi}\right]\hat{\boldsymbol{k}},\ \text{and}\\&\boldsymbol{M}_b=\left[M_0+\frac{K_1\beta z}{1+\chi}-\frac{K_2\beta' z}{1+\chi}\right]\hat{\boldsymbol{k}},\end{aligned} \tag{7}$$

where the subscript ‘b’ denotes the basic state.

We will analyze the stability of the basic state by introducing the perturbations

$$\begin{aligned} &\boldsymbol{q}=\boldsymbol{q}_{\mathrm{b}}+\boldsymbol{q}', \quad \rho=\rho_{\mathrm{b}}+\rho', \quad p=p_{\mathrm{b}}(z)+p', \\ &T=T_{\mathrm{b}}(z)+\theta, \quad C=C_{\mathrm{b}}(z)+\gamma, \\ &\boldsymbol{H}=\boldsymbol{H}_{\mathrm{b}}(z)+\boldsymbol{H}', \text{ and} \\ &\boldsymbol{M}=\boldsymbol{M}_{\mathrm{b}}(z)+\boldsymbol{M}', \end{aligned} \tag{8}$$

where $\boldsymbol{q}'=(u,v,w)$, ρ', p', θ, γ, $\boldsymbol{H}'=(H_1',H_2',H_3')$, and $\boldsymbol{M}'=(M_1',M_2',M_3')$ are perturbations in the velocity $\boldsymbol{q}$, density ρ, pressure p, temperature T, concentration C, magnetic field intensity $\boldsymbol{H}$, and magnetization $\boldsymbol{M}$, respectively.

The relevant nonlinear perturbation equations for a double-diffusive magnetized ferrofluid become

$$\begin{aligned} &\rho_0\frac{\partial \boldsymbol{q}'}{\partial t}=-\nabla p'+\mu\nabla^2\boldsymbol{q}'+\rho_0 g(\alpha\theta-\alpha'\gamma)\hat{\boldsymbol{k}} \\ &-\mu_0\left(\frac{K_1\beta}{1+\chi}-\frac{K_2\beta'}{1+\chi}\right)\left[(1+\chi)\phi_z'\hat{\boldsymbol{k}}-K_1\theta\hat{\boldsymbol{k}}+K_2\gamma\hat{\boldsymbol{k}}\right] \\ &-\rho_0\boldsymbol{q}'\cdot\nabla\boldsymbol{q}'+\mu_0\frac{M_0}{H_0}\phi_x'\nabla\phi_x'+\mu_0\frac{M_0}{H_0}\phi_y'\nabla\phi_y' \\ &+\mu_0\chi\phi_z'\nabla\phi_z'-\mu_0K_1\theta\nabla\phi_z'+\mu_0K_2\gamma\nabla\phi_z', \end{aligned} \tag{9}$$

$$\nabla\cdot\boldsymbol{q}'=0, \tag{10}$$

$$\frac{\partial\theta}{\partial t}+\boldsymbol{q}'\cdot\nabla\theta=\kappa\nabla^2\theta+\beta w, \tag{11}$$

$$\frac{\partial\gamma}{\partial t}+\boldsymbol{q}'\cdot\nabla\gamma=\kappa'\nabla^2\gamma+\beta' w, \tag{12}$$

$$\left(1+\frac{M_0}{H_0}\right)\nabla^2\phi'-\left(\frac{M_0}{H_0}-\chi\right)\phi_{zz}'=K_1\theta_z-K_2\gamma_z. \tag{13}$$

Here, $\boldsymbol{H}'=\nabla\phi'$ [by $(5)_2$], where ϕ' is the perturbed magnetic potential depending on the temperature and solute concentration. In the present problem, instability is driven by two sources, the temperature field and the salt field. Since temperature and solute are considered to have an independent impact on the problem, we can take the potential ϕ' as the difference of two potentials, ϕ_1' and ϕ_2', one analogous to the temperature and the other analogous to the solute. Further analysis will be carried out using these two potentials.

Equations (9)–(13) are non-dimensionalized (dropping*), to obtain

$$\begin{aligned} &\frac{\partial\boldsymbol{q}}{\partial t}=-\nabla p+\nabla^2\boldsymbol{q}+R^{1/2}(1+M_1-M_4)\theta\hat{\boldsymbol{k}} \\ &-\frac{S^{1/2}}{Le}(1+M_4'-M_1')\gamma\hat{\boldsymbol{k}}-R^{1/2}(M_1-M_4)\phi_{1z}\hat{\boldsymbol{k}} \\ &+\frac{S^{1/2}}{Le}(M_4'-M_1')\phi_{2z}\hat{\boldsymbol{k}}-M_1\theta\nabla\phi_{1z} \\ &+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\theta\nabla\phi_{2z}+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\gamma\nabla\phi_{1z}-\frac{M_1'}{Le}\gamma\nabla\phi_{2z} \\ &+\left(M_3-\frac{1}{1+\chi}\right)\Big[M_1\phi_{1x}\nabla\phi_{1x}-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\big(\phi_{1x}\nabla\phi_{2x} \\ &+\phi_{2x}\nabla\phi_{1x}\big)+\frac{M_1'}{Le}\phi_{2x}\nabla\phi_{2x}\Big]+\left(M_3-\frac{1}{1+\chi}\right)\Big[M_1\phi_{1y}\nabla\phi_{1y} \\ &-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\big(\phi_{1y}\nabla\phi_{2y}+\phi_{2y}\nabla\phi_{1y}\big)+\frac{M_1'}{Le}\phi_{2y}\nabla\phi_{2y}\Big] \\ &+\frac{\chi}{1+\chi}\Big[M_1\phi_{1z}\nabla\phi_{1z}-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\big(\phi_{1z}\nabla\phi_{2z} \\ &+\phi_{2z}\nabla\phi_{1z}\big)+\frac{M_1'}{Le}\phi_{2z}\nabla\phi_{2z}\Big]-\boldsymbol{q}\cdot\nabla\boldsymbol{q}, \end{aligned} \tag{14}$$

$$\nabla\cdot\boldsymbol{q}=0, \tag{15}$$

$$\frac{\partial\theta}{\partial t}+\boldsymbol{q}\cdot\nabla\theta=\nabla^2\theta+R^{1/2}w, \tag{16}$$

$$\frac{\partial\gamma}{\partial t}+\boldsymbol{q}\cdot\nabla\gamma=\frac{1}{Le}\nabla^2\gamma+S^{1/2}w, \tag{17}$$

$$M_3\nabla^2\phi_1-(M_3-1)\phi_{1zz}=\theta_z, \tag{18}$$

$$M_3\nabla^2\phi_2-(M_3-1)\phi_{2zz}=\gamma_z, \tag{19}$$

where the following non-dimensional quantities and non-dimensional parameters are introduced:

$$\begin{aligned} &t^*=\frac{\nu}{d^2}t, \quad \boldsymbol{q}^*=\frac{d}{\nu}\boldsymbol{q}'^*, \quad \theta^*=\frac{R^{1/2}}{\beta d}\theta, \\ &\gamma^*=\frac{S^{1/2}}{\beta' d}\gamma, \quad \phi_1^*=\frac{(1+\chi)R^{1/2}}{K_1\beta d^2}\phi_1', \\ &\phi_2^*=\frac{(1+\chi)S^{1/2}}{K_2\beta' d^2}\phi_2', \quad p^*=\frac{d^2}{\mu\kappa}p', \\ &z^*=\frac{1}{d}z, \quad R=\frac{g\alpha\beta\rho_0 d^4}{\mu\kappa}, \quad S=\frac{g\alpha'\beta'\rho_0 d^4}{\mu\kappa'}, \\ &M_1=\frac{\mu_0K_1^2\beta}{\alpha\rho_0 g(1+\chi)}, \quad M_1'=\frac{\mu_0K_2^2\beta'}{\alpha'\rho_0 g(1+\chi)}, \\ &N=RM_1=\frac{\mu_0K_1^2\beta^2d^4}{\mu\kappa(1+\chi)}, \quad M_3=\frac{\left(1+\frac{M_0}{H_0}\right)}{(1+\chi)}, \\ &M_4=\frac{\mu_0K_1K_2\beta'}{\alpha\rho_0 g(1+\chi)}, \quad M_4'=\frac{\mu_0K_1K_2\beta}{\alpha\rho_0 g(1+\chi)}, \\ &M_5=\frac{M_4}{M_1}=\frac{M_1'}{M_4'}=\frac{K_2\beta'}{K_1\beta}, \text{ and } Le=\frac{\kappa}{\kappa'}. \end{aligned}$$

Here, R is the thermal Rayleigh number, N the magnetic thermal Rayleigh number, S the salt Rayleigh number, M_1 the ratio of the magnetic to gravitational forces, and M_1' the effect of magnetization due to salinity. The parameter M_3 measures the departure of linearity in the magnetic equation of state; values from one ($M_0 = \chi H_0$) or higher are possible for the usual equations of state. M_5 represents the ratio of the salinity effect on the magnetic field to the pyromagnetic coefficient and Le is the Lewis number.

The boundary conditions are

$$\begin{aligned} &w=0, \quad u_z=0, \quad v_z=0, \quad \theta=0, \\ &\gamma=0, \quad \phi_{1z}'=0, \quad \phi_{2z}'=0, \quad \text{at} \quad z=\pm\frac{1}{2}, \end{aligned} \tag{20}$$

and $\boldsymbol{q}'$, θ, γ, ϕ' satisfy plane tiling periodicity.

In the next section, we develop a nonlinear energy stability analysis for (14)–(19).

3. Energy Stability Criterion with a Generalized Energy

We will suppose that the three-dimensional disturbance occupies a periodic cell V in the layer and denote $\|\cdot\|$ and $\langle\cdot\rangle$ as norm and inner product on $L^2(V)$. Then, we begin the nonlinear stability analysis by multiplying (14) by $\boldsymbol{q}$, (16) by θ, (17) by γ, (18) by ϕ_1, (19) by ϕ_2 and integrating over V to find [after using (15), the boundary conditions and the divergence theorem]

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}\|\boldsymbol{q}\|^2}{\mathrm{d}t} &= -\|\nabla\boldsymbol{q}\|^2 + R^{1/2}(1+M_1-M_4)\langle w\theta\rangle \\ &-\frac{S^{1/2}}{Le}(1+M_4'-M_1')\langle w\gamma\rangle - R^{1/2}(M_1-M_4)\langle w\phi_{1z}\rangle \\ &+\frac{S^{1/2}}{Le}(M_4'-M_1')\langle w\phi_{2z}\rangle - \frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\langle\boldsymbol{q}\cdot\nabla\theta\phi_{2z}\rangle \\ &-\frac{M_4^{1/2}M_4^{1/2}}{Le^{1/2}}\langle\boldsymbol{q}\cdot\nabla\gamma\phi_{1z}\rangle + \frac{M_1'}{Le}\langle\boldsymbol{q}\cdot\nabla\gamma\phi_{2z}\rangle \\ &+\Big(M_3-\frac{1}{1+\chi}\Big)M_1\langle\phi_{1x}\boldsymbol{q}\cdot\nabla\phi_{1x}\rangle + \Big(M_3-\frac{1}{1+\chi}\Big) \\ &\cdot\Big[-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(\langle\phi_{1x}\boldsymbol{q}\cdot\nabla\phi_{2x}\rangle + \langle\phi_{2x}\boldsymbol{q}\cdot\nabla\phi_{1x}\rangle) \\ &+\frac{M_1'}{Le}\langle\phi_{2x}\boldsymbol{q}\cdot\nabla\phi_{2x}\rangle\Big] + \Big(M_3-\frac{1}{1+\chi}\Big)\Big[M_1\langle\phi_{1y}\boldsymbol{q}\cdot\nabla\phi_{1y}\rangle \\ &-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(\langle\phi_{1y}\boldsymbol{q}\cdot\nabla\phi_{2y}\rangle + \langle\phi_{2y}\boldsymbol{q}\cdot\nabla\phi_{1y}\rangle)\Big] \\ &+\Big(M_3-\frac{1}{1+\chi}\Big)\frac{M_1'}{Le}\langle\phi_{2y}\boldsymbol{q}\cdot\nabla\phi_{2y}\rangle + \frac{\chi}{1+\chi}M_1\langle\phi_{1z}\boldsymbol{q}\cdot\nabla\phi_{1z}\rangle \\ &+M_1\langle\phi_{1z}\boldsymbol{q}\cdot\nabla\theta\rangle + \frac{\chi}{1+\chi}\Big[-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\Big(\langle\phi_{1z}\boldsymbol{q}\cdot\nabla\phi_{2z}\rangle \\ &+\langle\phi_{2z}\boldsymbol{q}\cdot\nabla\phi_{1z}\rangle\Big) + \frac{M_1'}{Le}\langle\phi_{2z}\boldsymbol{q}\cdot\nabla\phi_{2z}\rangle\Big], \end{aligned} \tag{21}$$

$$\frac{1}{2}\frac{\mathrm{d}\|\theta\|^2}{\mathrm{d}t} = -\|\nabla\theta\|^2 + R^{1/2}\langle w\theta\rangle, \tag{22}$$

$$\frac{1}{2}\frac{\mathrm{d}\|\gamma\|^2}{\mathrm{d}t} = -\frac{1}{Le}\|\nabla\gamma\|^2 + S^{1/2}\langle w\gamma\rangle, \tag{23}$$

$$M_3\|\nabla\phi_1\|^2 - (M_3-1)\|\phi_{1z}\|^2 + \langle\theta_z\phi_1\rangle = 0, \tag{24}$$

$$M_3\|\nabla\phi_1\|^2 - (M_3-1)\|\phi_{2z}\|^2 + \langle\gamma_z\phi_2\rangle = 0. \tag{25}$$

To study the nonlinear stability of the basic state (7), an L^2 energy, $E(t)$, is constructed using (21)–(25); the evolution of $E(t)$ is given by

$$\frac{\mathrm{d}E}{\mathrm{d}t} = I_0 - D_0 + N_0, \tag{26}$$

where

$$E = \frac{1}{2}\|\theta\|^2 + \frac{\lambda_1}{2}\|\boldsymbol{q}\|^2 - \frac{\lambda_3}{2}\|\gamma\|^2, \tag{27}$$

$$\begin{aligned} I_0 &= R^{1/2}\{1+\lambda_1(1+M_1-M_4)\}\langle w\theta\rangle \\ &- S^{1/2}\left\{\lambda_3 + \frac{\lambda_1}{Le}(1+M_4'-M_1')\right\}\langle w\gamma\rangle \\ &- R^{1/2}\lambda_1(M_1-M_4)\langle w\phi_{1z}\rangle \\ &+ \frac{S^{1/2}\lambda_1}{Le}(M_4'-M_1')\langle w\phi_{2z}\rangle - \lambda_2\langle\phi_1\theta_z\rangle + \lambda_4\langle\phi_2\gamma_z\rangle, \end{aligned} \tag{28}$$

$$\begin{aligned} D_0 &= \|\nabla\theta\|^2 + \lambda_1\|\nabla\boldsymbol{q}\|^2 - \frac{\lambda_3}{Le}\|\nabla\gamma\|^2 \\ &+ \lambda_2 M_3\|\nabla\phi_1\|^2 - \lambda_2(M_3-1)\|\phi_{1z}\|^2 \\ &- \lambda_4 M_3\|\nabla\phi_2\|^2 + \lambda_4(M_3-1)\|\phi_{2z}\|^2, \end{aligned} \tag{29}$$

$$\begin{aligned} N_0 &= \lambda_1 M_1\langle\boldsymbol{q}\cdot\nabla\theta\phi_{1z}\rangle - \frac{\lambda_1 M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\langle\boldsymbol{q}\cdot\nabla\theta\phi_{2z}\rangle \\ &-\frac{\lambda_1 M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\langle\boldsymbol{q}\cdot\nabla\gamma\phi_{1z}\rangle + \lambda_1\Big(M_3-\frac{1}{1+\chi}\Big) \\ &\cdot\Big[M_1\langle\phi_{1x}\boldsymbol{q}\cdot\nabla\phi_{1x}\rangle - \frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(\langle\phi_{1x}\boldsymbol{q}\cdot\nabla\phi_{2x}\rangle \\ &+\langle\phi_{2x}\boldsymbol{q}\cdot\nabla\phi_{1x}\rangle)\Big] + \lambda_1\Big(M_3-\frac{1}{1+\chi}\Big)\Big[\frac{M_1'}{Le}\langle\phi_{2x}\boldsymbol{q}\cdot\nabla\phi_{2x}\rangle \\ &+M_1\langle\phi_{1y}\boldsymbol{q}\cdot\nabla\phi_{1y}\rangle + \frac{M_1'}{Le}\langle\phi_{2y}\boldsymbol{q}\cdot\nabla\phi_{2y}\rangle\Big] \\ &+\lambda_1\frac{\chi}{\chi+1}\Big[-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(\langle\phi_{1z}\boldsymbol{q}\cdot\nabla\phi_{2z}\rangle + \langle\phi_{2z}\boldsymbol{q}\cdot\nabla\phi_{1z}\rangle) \\ &+\frac{M_1'}{Le}\langle\phi_{2z}\boldsymbol{q}\cdot\nabla\phi_{2z}\rangle\Big] + \lambda_1\frac{\chi}{\chi+1}M_1\langle\phi_{1z}\boldsymbol{q}\cdot\nabla\phi_{1z}\rangle \end{aligned}$$

$$+\frac{\lambda_1 M_1'}{Le}\langle \boldsymbol{q}\cdot\nabla\,\gamma\phi_{2z}\rangle-\frac{\lambda_1 M_4^{1/2}M_4'^{1/2}}{Le^{1/2}} \cdot\Big(M_3-\frac{1}{1+\chi}\Big)(\langle\phi_{1y}\boldsymbol{q}\cdot\nabla\,\phi_{2y}\rangle+\langle\phi_{2y}\boldsymbol{q}\cdot\nabla\,\phi_{1y}\rangle),\quad(30)$$

with λ_1, λ_2, λ_3, λ_4 being four positive coupling parameters.

Here, the negative sign of the term $\frac{\lambda_3}{2}\|\gamma\|^2$ in the energy equation (27) shows that the energy of the system is consumed due to the solute concentration as the system is soluted from below. Now, we take the assumption that the energy consumed due to the solute concentration is less than the energy produced due to velocity and temperature. We also assume that the energy dissipated by the solute concentration is less than the energy dissipated by the velocity, temperature and magnetization. These assumptions will ensure that all the terms on the right-hand side of (27) and (29) are always less than the left-hand side of that equation.

From (26), we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t}\le -a_0D_0+N_0,\quad(31)$$

with $a_0=1-m(>0)$, where

$$m=\max_H\frac{I_0}{D_0},\quad(32)$$

and H is the space of admissible solutions.

In order to dominate the nonlinear terms and for studying the (conditional) nonlinear stability, we now introduce the generalized energy functional as

$$V_{\mathrm{g}}(t)=E(t)+b_0E_1(t),\quad(33)$$

where b_0 is a positive coupling parameter to be chosen and the complementary energy $E_1(t)$ is given by

$$E_1(t)=\frac{1}{2}\|\nabla\theta\|^2+\frac{1}{2}\|\nabla\boldsymbol{q}\|^2+\frac{1}{2}\|\nabla\gamma\|^2.\quad(34)$$

The evolution of $V_{\mathrm{g}}(t)$ is given by

$$\frac{\mathrm{d}V_{\mathrm{g}}}{\mathrm{d}t}\le -a_0D_0+N_0+b_0I_1-b_0D_1+b_0N_1,\quad(35)$$

where

$$\begin{aligned}I_1&=R^{1/2}(2+M_1-M_4)\langle\nabla w\cdot\nabla\,\theta\rangle\\&+\frac{S^{1/2}}{Le}(Le-1+M_1'-M_4')\langle\nabla w\cdot\nabla\,\gamma\rangle\\&-R^{1/2}(M_1-M_4)\langle\nabla w\cdot\nabla\,\phi_{1z}\rangle\\&+\frac{S^{1/2}}{Le}(M_4'-M_1')\langle\nabla w\cdot\nabla\,\phi_{2z}\rangle,\end{aligned}\quad(36)$$

$$D_1=\|\nabla^2\boldsymbol{q}\|^2+\|\nabla^2\theta\|^2+\frac{1}{Le}\|\nabla^2\gamma\|^2,\quad(37)$$

$$\begin{aligned}N_1&=\langle\boldsymbol{q}\cdot\nabla\,\theta\,\nabla^2\theta\rangle+\langle\boldsymbol{q}\cdot\nabla\,\boldsymbol{q}\cdot\nabla^2\boldsymbol{q}\rangle+\langle\boldsymbol{q}\cdot\nabla\,\gamma\nabla^2\gamma\rangle\\&-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\langle\theta\,\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2z}\rangle+M_1\langle\theta\,\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1z}\rangle\\&-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\langle\gamma\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1z}\rangle+\frac{M_1'}{Le}\langle\gamma\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2z}\rangle\\&+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\Big(M_3-\frac{1}{1+\chi}\Big)[\langle\phi_{1x}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2x}\rangle\\&+\langle\phi_{2x}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1x}\rangle]-M_1\Big(M_3-\frac{1}{1+\chi}\Big)\langle\phi_{1x}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1x}\rangle\\&-\Big(\frac{\chi}{1+\chi}\Big)M_1\langle\phi_{1z}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1z}\rangle-\Big(M_3-\frac{1}{1+\chi}\Big)\\&\cdot\Big[\frac{M_1'}{Le}\langle\phi_{2x}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2x}\rangle+M_1\langle\phi_{1y}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1y}\rangle\\&+\frac{M_1'}{Le}\langle\phi_{2y}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2y}\rangle\Big]+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}\Big(M_3-\frac{1}{1+\chi}\Big)\\&\cdot[\langle\phi_{1y}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2y}\rangle+\langle\phi_{2y}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1y}\rangle]\\&-\frac{\chi}{1+\chi}\Big[-\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(\langle\phi_{1z}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2z}\rangle\\&+\langle\phi_{2z}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{1z}\rangle)+\frac{M_1'}{Le}\langle\phi_{2z}\nabla^2\boldsymbol{q}\cdot\nabla\,\phi_{2z}\rangle\Big].\end{aligned}\quad(38)$$

Now, we write some easily obtainable results from (18) and (19), and recall the embedding theorems

$$\begin{aligned}&\|\nabla\phi_1\|\le\|\theta\|,\quad\|\nabla\phi_{1z}\|\le\|\nabla^2\phi_1\|\le\|\nabla\theta\|,\\&\|\nabla^2\phi_{1z}\|\le\|\nabla^2\theta\|,\quad\|\nabla^2\phi_{1x}\|\le\|\nabla^2\theta\|,\\&\|\nabla^2\phi_{1y}\|\le\|\nabla^2\theta\|,\quad\|\nabla\phi_2\|\le\|\gamma\|,\\&\|\nabla\phi_{2z}\|\le\|\nabla^2\phi_2\|\le\|\nabla\gamma\|,\quad\|\nabla^2\phi_{2z}\|\le\|\nabla^2\gamma\|,\\&\|\nabla^2\phi_{2x}\|\le\|\nabla^2\gamma\|,\quad\|\nabla^2\phi_{2y}\|\le\|\nabla^2\gamma\|,\\&\|\nabla w\|\le\|\nabla\boldsymbol{q}\|,\quad\sup|F|\le C^*\|\nabla^2F\|,\\&F\in\{\boldsymbol{q},\theta,\gamma,\phi_{1x},\phi_{1y},\phi_{1z},\phi_{2x},\phi_{2y},\phi_{2z}\},\end{aligned}\quad(39)$$

where C^* is a computable positive constant depending on V; its value is given by Galdi and Straughan [38] and the statement is proved by Adams [39].

Therefore, by (36) using (29), (37), (39), the Cauchy-Schwartz and the Young inequalities [40], we have

$$\begin{aligned}b_0I_1\le 2b_0\varepsilon_0^2D_1&+\frac{b_0}{2\varepsilon_0^2\pi^2}\Big[R\{(2+M_1-M_4)^2\\&+(M_1-M_4)^2\}+\frac{S}{Le\lambda_3}\{(Le-1+M_1'-M_4')^2\\&+(M_4'-M_1')^2\}\Big]D_0.\end{aligned}$$

Choosing

$$\varepsilon_0^2 = \frac{1}{4}$$

and

$$b_0 = \frac{a_0\pi^2}{4}\Big[R\{(2+M_1-M_4)^2\}+(M_1-M_4)^2 + \frac{S}{Le\lambda_3}\{(Le-1+M_1'-M_4')^2+(M_4'-M_1')^2\}\Big]^{-1},$$

and defining

$$D_2 = \frac{a_0}{2}D_0 + \frac{b_0}{2}D_1, \tag{40}$$

it then follows easily that

$$b_0 I_1 \le D_2. \tag{41}$$

We next estimate the nonlinear terms N_1 and N_0. With the help of (33), (39) and (40) we find

$$N_1 \le C^*\left(\frac{2}{b_0}\right)^{3/2}\left[2+Le^{1/2}+\left(M_1+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(1+Le^{1/2})+\frac{M_1'}{Le^{1/2}}\right)\left(2M_3+\frac{2\chi-1}{1+\chi}\right)\right]D_2V_{\mathrm{g}}^{1/2}, \tag{42}$$

$$N_0 \le 2C^*\lambda_1\left(\frac{2}{b_0a_0}\right)^{1/2}\left(M_1+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}\lambda_3^{1/2}}(1+Le^{1/2})+\frac{M_1'}{Le^{1/2}\lambda_3}\right)\left(2M_3+\frac{2\chi-1}{1+\chi}\right)D_2V_{\mathrm{g}}^{1/2}. \tag{43}$$

Using (40), (41), (42) and (43) in (35), we get

$$\dot{V}_g(t) \le -D_2(1-\grave{A}V_{\mathrm{g}}^{1/2}), \tag{44}$$

where

$$\grave{A} = 2C^*\lambda_1\left(\frac{2}{b_0a_0}\right)^{1/2}\left(M_1+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}\lambda_3^{1/2}}(1+Le^{1/2})+\frac{M_1'}{Le^{1/2}\lambda_3}\right)\left(2M_3+\frac{2\chi-1}{1+\chi}\right) + 2C^*\left(\frac{2}{b_0}\right)^{1/2}\left[2+Le^{1/2}+\left(M_1+\frac{M_4^{1/2}M_4'^{1/2}}{Le^{1/2}}(1+Le^{1/2})+\frac{M_1'}{Le^{1/2}}\right)\left(2M_3+\frac{2\chi-1}{1+\chi}\right)\right]. \tag{45}$$

This last estimate enables us to prove the following theorem of conditional nonlinear stability criterion.

Theorem. *Let*

$$0 < m < 1, \tag{46}$$

$$V_{\mathrm{g}}(0) < \grave{A}^{-2}, \tag{47}$$

with $\grave{A}$ *given by (45). Then, there exists a positive constant* K^**, such that*

$$V_{\mathrm{g}}(t) \le V_{\mathrm{g}}(0)\mathrm{e}^{-K^*(1-\grave{A}V_{\mathrm{g}}^{1/2}(0))t}, \quad t \ge 0. \tag{48}$$

Proof. The hypothesis and inequality (44) ensures that

$$\dot{V}_{\mathrm{g}}(0) < 0.$$

Therefore, from inequality (44) by a recursive argument, we obtain

$$\dot{V}_{\mathrm{g}}(t) \le -D_2(1-\grave{A}V_{\mathrm{g}}^{1/2}(0)), \quad \forall \quad t \ge 0. \tag{49}$$

Now, we prove that there exists $K^* > 0$, such that

$$-D_2 \le -K^*V_{\mathrm{g}}. \tag{50}$$

By (33), using (29), (37) and (40), and by virtue of the Poincaré-type inequalities, we have

$$V_{\mathrm{g}} \le \frac{(2+Le)}{\pi^2}\left[\frac{D_0}{2}+\frac{b_0}{2}D_1\right] \le \frac{(2+Le)}{\pi^2}\left[1+\frac{m}{1-m}\right]D_2.$$

Let $k_0 > 0$, such that

$$k_0 \ge \frac{m}{1-m}, \tag{51}$$

then we obtain $V_{\mathrm{g}} \le \frac{1}{\pi^2}(2+Le)(1+k_0)D_2$. Then assuming $K^* = \frac{\pi^2}{(2+Le)(k_0+1)}$, with k_0 given by (51), from (49) and (50), we have

$$\dot{V}_{\mathrm{g}} \le -K^*V_{\mathrm{g}}(1-\grave{A}V_{\mathrm{g}}^{1/2}(0)), \quad \forall \quad t \ge 0. \tag{52}$$

Integrating this last inequality, we deduce the theorem.

Since $V_{\mathrm{g}}(t)$ in (33) does not contain the kinetic energy terms for the magnetic potential, $\|\nabla\phi_1\|^2$ and $\|\nabla\phi_2\|^2$, it is worthwhile to check what happens to $\|\nabla\phi_1\|^2$ and $\|\nabla\phi_2\|^2$ as $t\to\infty$.

Using inequality $(39)_1$ and $(39)_6$, we have

$$\|\nabla\phi_1\|^2 \le \|\theta\|^2 \text{ and } \|\nabla\phi_2\|^2 \le \|\gamma\|^2. \tag{53}$$

Thus, (33) and inequalities (53) ensure the decay of $\|\nabla\phi_1\|^2$ and $\|\nabla\phi_2\|^2$, i. e. $\|\boldsymbol{H}'\|^2$.

4. Variational Problem and the Rayleigh Number

We now solve the variational problem associated with the maximum problem (32). The corresponding Euler-Lagrange equations for the problem (32) at the critical argument $m=1$, after taking transformations $\hat{\boldsymbol{q}}=\sqrt{\lambda_1}\boldsymbol{q}$, $\hat{\phi}_1=\sqrt{\lambda_2}\phi_1$, $\hat{\gamma}=\sqrt{\lambda_3}\gamma$, and $\hat{\phi}_2=\sqrt{\lambda_4}\phi_2$ (dropping caps), are

$$\begin{aligned}&2\nabla^2\boldsymbol{q}+R^{1/2}\{1+\lambda_1(1+M_1-M_4)\}\frac{1}{\lambda_1^{1/2}}\theta\hat{\boldsymbol{k}}\\&-S^{1/2}\left\{\lambda_3+\frac{\lambda_1}{Le}(1+M_4'-M_1')\right\}\frac{1}{\lambda_1^{1/2}\lambda_3^{1/2}}\gamma\hat{\boldsymbol{k}}\\&-\frac{\lambda_1^{1/2}}{\lambda_2^{1/2}}R^{1/2}(M_1-M_4)\phi_{1z}\hat{\boldsymbol{k}}\\&+\frac{\lambda_1^{1/2}}{\lambda_4^{1/2}}\frac{S^{1/2}}{Le}(M_4'-M_1')\phi_{2z}\hat{\boldsymbol{k}}-2\nabla p=\boldsymbol{0},\end{aligned}\tag{54}$$

$$\begin{aligned}&2\nabla^2\theta+R^{1/2}\{1+\lambda_1(1+M_1-M_4)\}\frac{1}{\lambda_1^{1/2}}w\\&+\lambda_2^{1/2}\phi_{1z}=0,\end{aligned}\tag{55}$$

$$\begin{aligned}&\frac{2}{Le}\nabla^2\gamma+S^{1/2}\left\{\lambda_3+\frac{\lambda_1}{Le}(1+M_4'-M_1')\right\}\\&\cdot\frac{1}{\lambda_1^{1/2}\lambda_3^{1/2}}w+\frac{\lambda_4^{1/2}}{\lambda_3^{1/2}}\phi_{2z}=0,\end{aligned}\tag{56}$$

$$\begin{aligned}&2M_3\nabla^2\phi_1-2(M_3-1)\phi_{1zz}\\&+\frac{R^{1/2}\lambda_1^{1/2}}{\lambda_2^{1/2}}(M_1-M_4)w_z-\lambda_2^{1/2}\theta_z=0,\end{aligned}\tag{57}$$

$$\begin{aligned}&2M_3\nabla^2\phi_2-2(M_3-1)\phi_{2zz}\\&+\frac{S^{1/2}\lambda_1^{1/2}}{\lambda_4^{1/2}Le}(M_4'-M_1')w_z-\frac{\lambda_4^{1/2}}{\lambda_3^{1/2}}\gamma_z=0,\end{aligned}\tag{58}$$

where p is a Lagrange multiplier introduced, since $\boldsymbol{q}$ is solenoidal.

On taking the third component of the curl curl of (54), we find

$$\begin{aligned}&2\nabla^4w+R^{1/2}\{1+\lambda_1(1+M_1-M_4)\}\frac{1}{\lambda_1^{1/2}}\nabla_1^2\theta\\&-S^{1/2}\left\{\lambda_3+\frac{\lambda_1}{Le}(1+M_4'-M_1')\right\}\frac{1}{\lambda_1^{1/2}\lambda_3^{1/2}}\nabla_1^2\gamma\\&-\frac{\lambda_1^{1/2}}{\lambda_2^{1/2}}R^{1/2}(M_1-M_4)\nabla_1^2\phi_{1z}\\&+\frac{\lambda_1^{1/2}}{\lambda_4^{1/2}}\frac{S^{1/2}}{Le}(M_4'-M_1')\nabla_1^2\phi_{2z}=0.\end{aligned}\tag{59}$$

Now, we assume a plane tiling form

$$\begin{aligned}&(w,\theta,\gamma,\phi_1,\phi_2)\\&=[W(z),\Theta(z),\Gamma(z),\Phi_1(z),\Phi_2(z)]g(x,y),\end{aligned}\tag{60}$$

where $\nabla_1^2g+a^2g=0$, a being the wave number ([41], pp. 106 – 114; [13]). The wave number is found a posteriori to be non-zero; so from (55) – (59) we see that W, Θ, Γ, Φ_1, Φ_2 satisfy

$$\begin{aligned}&2(D^2-a^2)^2W-a^2R^{1/2}\{1+\lambda_1(1+M_1-M_4)\}\frac{1}{\lambda_1^{1/2}}\Theta\\&+\frac{a^2S^{1/2}}{\lambda_1^{1/2}\lambda_3^{1/2}}\left\{\lambda_3+\frac{\lambda_1}{Le}(1+M_4'-M_1')\right\}\Gamma\\&+\frac{\lambda_1^{1/2}}{\lambda_2^{1/2}}a^2R^{1/2}(M_1-M_4)D\Phi_1\\&-\frac{\lambda_1^{1/2}}{\lambda_4^{1/2}}\frac{a^2S^{1/2}}{Le}(M_4'-M_1')D\Phi_2=0,\end{aligned}\tag{61}$$

$$\begin{aligned}&2(D^2-a^2)\Theta+R^{1/2}\{1+\lambda_1(1+M_1\\&-M_4)\}\frac{1}{\lambda_1^{1/2}}W+\lambda_2^{1/2}D\Phi_1=0,\end{aligned}\tag{62}$$

$$\begin{aligned}&\frac{2}{Le}(D^2-a^2)\Gamma+\frac{S^{1/2}}{\lambda_1^{1/2}\lambda_3^{1/2}}\left\{\lambda_3+\frac{\lambda_1}{Le}(1+M_4'\right.\\&\left.-M_1')\right\}W+\frac{\lambda_4^{1/2}}{\lambda_3^{1/2}}D\Phi_2=0,\end{aligned}\tag{63}$$

$$\begin{aligned}&2(D^2-a^2M_3)\Phi_1+\frac{R^{1/2}\lambda_1^{1/2}}{\lambda_2^{1/2}}(M_1-M_4)DW\\&-\lambda_2^{1/2}D\Theta=0,\end{aligned}\tag{64}$$

$$2(D^2 - a^2 M_3)\Phi_2 - \frac{S^{1/2}\lambda_1^{1/2}}{\lambda_4^{1/2} Le}(M_4' - M_1')DW - \frac{\lambda_4^{1/2}}{\lambda_3^{1/2}} D\Gamma = 0. \tag{65}$$

The boundary conditions are

$$W = 0, \quad D^2 W = 0, \quad \Theta = 0, \quad \Gamma = 0, \quad D\Phi_1 = 0, \quad D\Phi_2 = 0 \text{ at } z = \pm\frac{1}{2}. \tag{66}$$

The exact solution subject to these boundary conditions is written in the form

$$\begin{aligned} &W = A_1 \cos\pi z, \quad \Theta = A_2 \cos\pi z, \quad D\Phi_1 = A_3 \cos\pi z, \quad \Phi_1 = \frac{A_3}{\pi}\sin\pi z, \quad \Gamma = A_4 \cos\pi z, \\ &D\Phi_2 = A_5 \cos\pi z, \quad \Phi_2 = \frac{A_5}{\pi}\sin\pi z, \end{aligned} \tag{67}$$

where A_1, A_2, A_3, A_4, A_5 are constants. Substituting solution (67) in (61)–(65), we get the equations involving coefficients of A_1, A_2, A_3, A_4, A_5. For the existence of non-trivial solutions, the determinant of the coefficients of A_1, A_2, A_3, A_4, A_5 must vanish. This determinant on simplification yields

$$R_e = \max_{\lambda_1,\lambda_2',\lambda_3,\lambda_4'} \min_x R_e'(\lambda_1, \lambda_2', \lambda_3, \lambda_4', x, M_1, M_3, M_1', M_5, Le, S_1), \tag{68}$$

where $R_e' = \frac{R}{\pi^4}$, $S_1 = \frac{S}{\pi^4}$, $x = \frac{a^2}{\pi^2}$, $\lambda_2' = \frac{\lambda_2}{\pi^2}$, $\lambda_4' = \frac{\lambda_4}{\pi^2}$.

Optimal values of λ_1, λ_2', λ_3 and λ_4' are determined by the conditions $\frac{dR_e}{d\lambda_1} = 0$, $\frac{dR_e}{d\lambda_2'} = 0$, $\frac{dR_e}{d\lambda_3} = 0$, $\frac{dR_e}{d\lambda_4'} = 0$, respectively, and are found to be

$$\lambda_1 = \frac{1}{1 + M_1(1 - M_5)}, \lambda_2' = \frac{(1+x)M_1(1-M_5)}{1 + M_1(1 - M_5)}, \lambda_3 = \frac{1 + M_1'\left(\frac{1}{M_5} - 1\right)}{Le[1 + M_1(1 - M_5)]}, \lambda_4' = \frac{(1+x)M_1'\left(\frac{1}{M_5} - 1\right)}{Le^2[1 + M_1(1 - M_5)]}. \tag{69}$$

Substituting (69) in (68), we have

$$R_e = \frac{\left\{4(1 + xM_3) - \frac{M_1(1-M_5)}{1+M_1(1-M_5)}\right\}\left[(1+x)^3 + xS_1\left\{1 + M_1'\left(\frac{1}{M_5} - 1\right)\right\}\right]}{x\{4(1 + xM_3)\{1 + M_1(1 - M_5)\} - 2M_1(1 - M_5)\}}. \tag{70}$$

If M_1 is sufficiently large, we obtain the magnetic thermal Rayleigh number

$$N_e = M_1 R_e = \frac{(3 + 4xM_3)\left[(1+x)^3 + xS_1\left\{1 + M_1'\left(\frac{1}{M_5} - 1\right)\right\}\right]}{x(2 + 4xM_3)(1 - M_5)}. \tag{71}$$

As a function of x, N_e given by (71) attains its minimum, if

$$P_5 x^5 + P_4 x^4 + P_3 x^3 + P_2 x^2 + P_1 x + P_0 = 0. \tag{72}$$

The coefficients $P_0 \ldots P_5$, being quite lengthy, are not written here and are evaluated during numerical calculations. The Newton-Raphson method is used to determine the values of the critical wave number in nonlinear stability results by the condition $\frac{dN_e}{dx} = 0$. With x determined as a solution of (72), (71) will give the required critical magnetic thermal Rayleigh number, N_{ce}.

In the absence of solute, (71) reduces to $N_e = (1+x)^3(3 + 4xM_3)/x(2 + 4xM_3)$, which is in good agreement with previous published results [27].

For the fixed values $S_1 = 100$, $M_1' = 0.1$, $M_5 = 0.1$, the critical wave number, x_{ce}, and the critical magnetic thermal Rayleigh number, N_{ce}, depend on the parameter M_3, taking the values

$$N_{ce} = 248.69, \quad x_{ce} = 2.09 \quad \text{for} \quad M_3 = 1, \quad N_{ce} = 218.61, \quad x_{ce} = 0.5 \quad \text{for} \quad M_3 \to \infty,$$

and intermediate values for intermediate M_3. Here, we can rearrange (70) to demonstrate the interaction of the buoyancy and magnetic modes of instability:

$$\frac{R_e}{R_{ce}} + \frac{N_e}{N_{ce}} \left\{ \frac{N_{ce}}{106.75} \left[\frac{(2+4M_3x)(1-M_5)}{3+4M_3x} \right] \right\} = \frac{(1+x)^3 + xS_1 \left\{ 1 + M_1' \left(\frac{1}{M_5} - 1 \right) \right\}}{x(106.75)}. \tag{73}$$

When M_3 is very large, (73) reduces to $R_e/R_{ce} + (N_e/N_{ce})(1.84) = 1.84$.

Here, we remark that in the absence of a solute gradient, there is tight coupling between the buoyancy and magnetic forces in the nonlinear energy stability analysis [27], whereas in the presence of a solute gradient, each individual convective mechanism yields the different wave number, so tight coupling is not possible in the present case.

For analyzing the linear instability results, we return to the non-dimensional equations (14) – (19), neglecting the nonlinear terms. We again perform the standard, stationary, mode analysis and look for the solution of these equations in the form (60). The boundary conditions in the present case are the same, i. e. (66). Following the procedure stated earlier in the energy stability case, we have

$$R_\ell = \frac{(1+x)^3(1+xM_3) + xS_1 \left[1 + xM_3 + xM_1'M_3 \left(\frac{1}{M_5} - 1 \right) \right]}{x[1 + xM_3 + xM_1M_3(1-M_5)]}. \tag{74}$$

We again consider that the magnetic thermal Rayleigh number, N_ℓ, depends on the parameter M_3. For sufficiently large M_1, the critical magnetic thermal Rayleigh number, in linear case, is

$$N_\ell = \frac{(1+x)^3(1+xM_3) + xS_1 \left[1 + xM_3 + xM_1'M_3 \left(\frac{1}{M_5} - 1 \right) \right]}{x^2M_3(1-M_5)}. \tag{75}$$

In the absence of solute, (75) reduces to $N_\ell = (1+x)^3(1+xM_3)/x^2M_3$, which is in good agreement with the previously published work [17].

There are instances in which the two theories coincide. This is true for the classical Bénard problem. In the absence of magnetic parameters ($M_1 = 0$, $M_1' = 0$ and $M_3 = 0$), we obtain

$$R_\ell = \frac{(1+x)^3}{x} + S_1 = R_e.$$

In the absence of solute ($S_1 = 0$), this further simplifies to $R_\ell = \frac{(1+x)^3}{x} = R_e$, i. e., in both the cases, the linear instability boundary $\equiv$ the nonlinear stability boundary.

Here, the energy method leads to the strong result that arbitrary subcritical instabilities are not possible, which is in good agreement with previously published works [3, 30]. Thus, for lower values of the magnetic parameters, this coincidence is immediately lost.

5. Results and Conclusion

The critical wave numbers, $x_{c\ell}$, x_{ce}, and critical magnetic thermal Rayleigh numbers, $N_{c\ell}$, N_{ce}, depend on M_3, S_1, M_1' and M_5. The variation of $x_{c\ell}$, x_{ce} and $N_{c\ell}$, N_{ce} with various parameters are given in Tables 1 and 2, and the results are further illustrated in Figs. 1 and 2.

Figure 1 presents the plot of the critical magnetic thermal Rayleigh numbers, $N_{c\ell}$ and N_{ce} versus the magnetic parameter M_3. This figure indicates that the magnetic parameter M_3 has a destabilizing effect because, as M_3 increases, the values of $N_{c\ell}$ and N_{ce} decrease. We also note that the values of $N_{c\ell}$ are always higher than those of N_{ce}, and this is quite un-

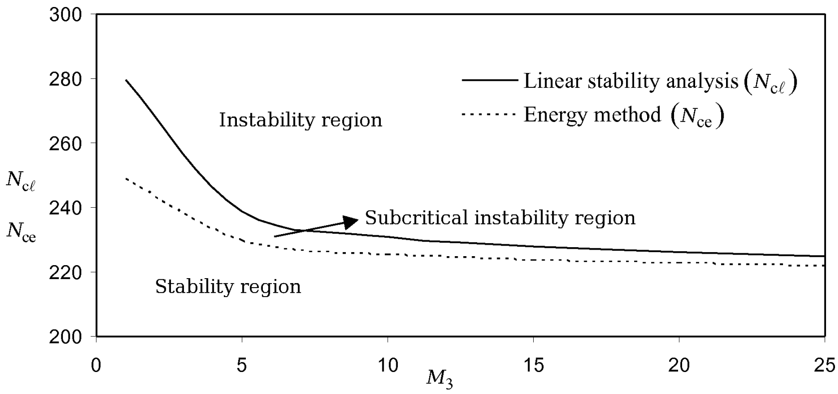

Fig. 1. The variation of the critical magnetic thermal Rayleigh numbers ($N_{c\ell}$ and N_{ce}) with the magnetic parameter (M_3) for $S_1 = 100$, $M_1' = 0.1$, $M_5 = 0.1$.

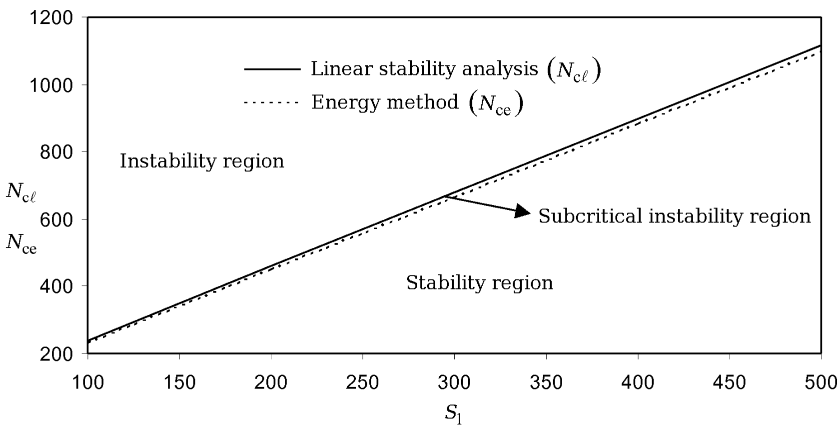

Fig. 2. The variation of the critical magnetic thermal Rayleigh numbers ($N_{c\ell}$ and N_{ce}) with the solute gradient (S_1) for $M_3 = 5$, $M_1' = 0.1$, $M_5 = 0.1$.

Table 1. The variation of the critical magnetic thermal Rayleigh numbers ($N_{c\ell}$ and N_{ce}) with the magnetic parameter (M_3) for $S_1 = 100$, $M_1' = 0.1$, $M_5 = 0.1$.

M_3	$x_{c\ell}$	$N_{c\ell}$	x_{ce}	N_{ce}	$N_{c\ell} - N_{ce}$
1	3.18	279.60	2.09	248.69	30.91
5	1.79	238.51	1.29	229.42	9.09
10	1.39	230.80	1.03	225.23	5.57
15	1.21	227.69	0.91	223.50	4.19
20	1.09	225.94	0.84	222.52	3.42
25	1.01	224.80	0.79	221.88	2.92
∞	0.5	218.61	0.5	218.61	0

derstandable because the linear stability theory gives sufficient conditions for instability, while the energy stability theory gives the sufficient condition for stability. Thus, the difference between the values of $N_{c\ell}$ and N_{ce} reveals that there is a band of Rayleigh numbers where subcritical instabilities may arise. We note that this band decreases as M_3 increases (Table 1). Figure 2 presents the plot of the critical magnetic thermal Rayleigh numbers, $N_{c\ell}$ and N_{ce}, versus the solute gradient S_1. This figure indicates that the solute gradient S_1 has a stabilizing effect because, with the increase of S_1, the values of $N_{c\ell}$ and N_{ce} also increase. We note that the subcritical instability region expands with the increase of the solute gradient S_1 (Table 2).

Table 2. The variation of the critical magnetic thermal Rayleigh numbers ($N_{c\ell}$ and N_{ce}) with the solute gradient (S_1) for $M_3 = 5$, $M_1' = 0.1$, $M_5 = 0.1$.

S_1	$x_{c\ell}$	$N_{c\ell}$	x_{ce}	N_{ce}	$N_{c\ell} - N_{ce}$
100	1.79	238.51	1.29	229.42	9.09
200	2.31	460.42	1.68	447.17	13.25
300	2.68	680.41	1.97	663.74	16.67
400	2.99	899.36	2.20	879.71	19.65
500	3.24	1117.60	2.39	1095.23	22.37

In conclusion, for the proposed model, we are able to derive a rigorous nonlinear energy stability result for a magnetized ferrofluid by performing a nonlinear energy stability (conditional) analysis. We derive a nonlinear stability threshold very close to the linear instability one. We also see that the magnetic mechanism alone can induce a subcritical region of instability. For the convection problem in magnetized ferrofluids, the linear critical magnetic thermal Rayleigh number is found to be higher than the nonlinear (energy) critical magnetic thermal Rayleigh number, which shows the possibility of the existence of subcritical instability. It is important to realize that this region decreases as magnetization increases. We also observe that a solute gradient cannot induce a subcritical region of instability, but concerning a magnetic mechanism, this region expands with the increase of the solute gradient.

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